Magnetic Dipole Torque Theorem Proof for Polar Curves

Raven Asher Raziel
12/10/02024

Prelude

If the textbook only stated that for a rectangular loop or a circular loop, the torque formula involving the area for a magnetic dipole moment would have been enough. However, by stating that it works for any arbitrary loop without any proof or any indication of how it would be proved (for the general case), my curiosity had become possessed.

There is no proof for the general formula online. If there was, I would not have spent so much time on and off on this question; although, I would not have grown either. This problem was so complicated and obscure that symbolab could not help with double-checking some of the lemmas, nor did I trust chat gpt to guide me (I personally do not trust it in general).

I cite Wikipedia, Paul's Online Math Notes, and Khan Academy as a source of various advanced trigonometric, calculus, and linear algebra theorems.

Now, the torque of a wire could be viewed as only proven empirically to be considered physical "law"; however, it must still be in agreement with the equation for the force of a current on a wire. Fortunately, this was not an approximation leading to a wild goose chase or a deadly doomscroll. And as I worked from the rectangular and circular case, it did point to the potential of the general function involving the area itself.

Note on the pedantic style: *When I first began to interpret proofs, in higher math, it is assumed the student knows certain things, thus for brevity, it is skipped over. Often, I would have to pause to complete these "exercises left for the reader"; however, I wanted to convey how my mind flowed the symbols and give those who are new to this something that seems daunting but within reach, so they are encouraged to pursue larger dragons. And perhaps forge their own.*

This proof also forgoes certain symbolic shortcuts that are often used. Whenever these shortcuts are thrown after it brings back the feeling of confusing and then resorting to symbol pushing. Math is certain, and although it can be advantageous to forgo overly complicated proofs, presenting symbolic tricks as "it's a dx or a Riemann sum" after telling the student dx is just part of the integration symbol leads to a nasty feeling. As long as they understand it's a symbolic trick and know of the origin for it, it is fine, but that is often not the case. So this proof, due to its breadth shall sacrifice some ink.

Finally, the style of proof is a bit more pedantic than most texts; however, given the monstrosity this is. No stone is left unturned so those who are studying proofs and mathematical methods can see it in full as an example of "embracing the complexity" (Professor Vance, ACC).

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Introduction to Torque on a Magnetic Loop

In many text books, the torque for a single loop of wire can be given by $\tau = \vec{u} \times \vec{B}$ where $\vec{u} = IA\,\hat{n}$. A is the area of the loop. The vector **n** is the unit vector given by the right hand rule if the fingers were curling in the direction of current and the thumb points up.

It states that the formula is valid for not only a rectangular loop or a circular loop, but any plane coil of an arbitrary shape. Often, a limited proof that assumes one of the sides being along axis of rotation is provided. The amount of vectors (4 to 5) is already daunting. The proof for the circular loop requires a triple vector product expansion, but only takes up a page.

More formally, if the loop is described by a vector polar equation $\vec{r}(\theta)$ on some plane, then the normal vector is:

$$\hat{n} = \frac{1}{\left|r(\vec{\theta})\right|}\vec{r}(\theta) \times \frac{1}{\left|\hat{r}\,'(\theta)\right|}\hat{r}\,'(\theta)$$

The equation for the total torque on a wire in a constant magnetic field is:

$$\tau = \int_0^L \vec{r}(s) \times (I\,\hat{l}(s) \times \vec{B})\,ds$$

Intuitively, the formula is adding the torque from every vector along the direction $\hat{l}(s)$ of infinitely small magnitude “ds” that composes a curve C of length L.

Enough infinitesimal calculus.

For a closed loop:

$$\oint_C \vec{r} \times (I\,\hat{l} \times \vec{B})\,ds$$

According to the Magnetic Dipole Moment, the torque is related directly not the length of the wire but the area that it encloses. The proof for a rectangular loop and a circular loop are relatively simple. However, let us find a general proof.

The Proof

Assume that s can be expressed as a function of angle: $s(\theta)$

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$$\tau = \int_0^L \vec{r}(s) \times (I\,\hat{l}(s) \times \vec{B})\,ds$$

The function is being used to describe a closed loop going around, thus, a polar form is more advantageous. To prove the theorem for a wire being described by a polar equation, the formula for the area of a polar equation from 0 to 2π must emerge:

$$A = \int_0^{2\pi} \frac{r(\theta)^2}{2}\,ds$$

Change of Variable

The plan will be to convert the function into terms of θ. There is a connection between arc length and angle.

$$s = \int_0^{\theta} \sqrt{r(t)^2 + \left(\frac{dr}{dt}\right)^2}\,dt$$

First fundamental theorem of Calculus

$$\frac{ds}{d\theta} = \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}$$

The change of variable is valid, because the function that is integrated is a function of t, and becomes a function of . Since t is a placeholder variable, the relationship between of t and θ is siPreludemply t = θ.

Utilizing the theorems for inverse functions and their derivatives, an important derivative is derived. Another constraint is that the change of length with respect to the angle cannot be 0.

$$\frac{1}{\frac{ds}{d\theta}} = \frac{1}{\sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}}$$

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}}$$

$$\frac{d\theta}{ds}\sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} = 1$$

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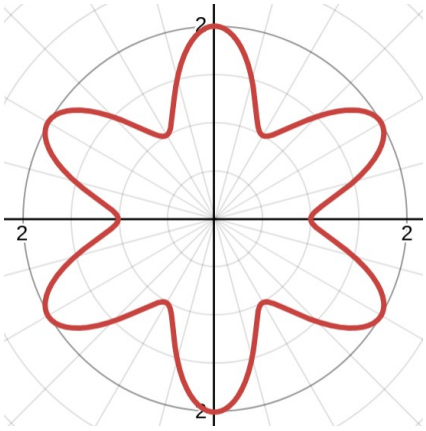
Most presentations of the change of variable rule solve for the "differential". This is a symbolic trick, even Calculus I by Tom Apostol teaches it (as a symbolic trick). But it is already taught in non-infinitesimal calculus that the "dx" is just part of the symbol of the operation of integration. Like the "x" of multiplication or the cross product, it is not a variable itself. So, this method of notation is utilized..

Before this procedure is carried out, note that L, is the length of the wire as it goes around in a single complete turn. If the function being integrated was a function of angle, then it would go from 0 to 2π.

S and angle are related by a one to one function. If not, then the loop would sweep over an area multiple times, which would bring into question the "area enclosed". Perhaps a formulation is valid (or not) when theta and s are non-monotonic; although, that will be "left as an exercise to the reader".



$$\tau = \int_0^L \vec{r}(s) \times (I\,\hat{l}(s) \times \vec{B})\, ds$$

$$\frac{d\theta}{ds}\sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} = 1$$

$$\tau = \int_0^L \vec{r}(s) \times (I\,\hat{l}(s) \times \vec{B}) \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} \frac{d\theta}{ds}\, ds$$

$$\tau = \int_{\theta(0)}^{\theta(L)} \vec{r}(s) \times (I\,\hat{l}(s) \times \vec{B}) \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2} \frac{d\theta}{ds}\, ds$$

$$\tau = \int_0^{2\pi} \vec{r}(\theta) \times (I\,\hat{l}(\theta) \times \vec{B}) \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

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Expression in a Favorable Basis

Although the integral is with respect to angle, there is a rather complicated term. In order to simplify, we must discuss the specifics of the vector **r** and **B**.

The coil must lie on a plane in order to meaningfully enclose an “area”. One method is to simply restrict the coil to the xy plane, yet that is insufficient to describe general situations. Furthermore, if the theorem were to describe the motion of the coil as a function of time as it rotated in 3D space, it cannot be restricted to the xy plane.

Yet, this method of simplification is easy to generalize. As much as I find beauty within mathematical notations across various fields, we shall utilize linear combinations of vectors instead of the notation for change of basis. This is also my first time using latex for rendering math formulae.

The only requirement is that the coil must lie on a plane. Any point on this plane can be represented by a linear combination of two vectors that are linearly independent. In this proof, they will be unit vectors that are orthogonal. The symbol θ will denote them, because they are essentially the **i** and **j** under a rotation. They need not be derived from those vectors, but it can be proven that there exists a linear transformation of rotation that transla{ { - 1 ^ 2 + { 1 over 2 } } 1 ^ 4 }te **i** and **j** to those vectors. (Look up the linear transformation on Wikipedia, it will frighten you.)

$\hat{i}_\theta \perp \hat{j}_\theta$

In order to complete the alternative coordinate system:

$\hat{k}_\theta = \hat{i}_\theta \times \hat{j}_\theta$

I will invoke another *magical mathematical phrase*: **it can be seen from mere observation** that all rules of the conventional coordinate system will apply with these new vectors.

$\vec{r}(\theta) = r(\theta)\cos(\theta)\vec{i_\theta} + r(\theta)\sin(\theta)\vec{j}_\theta$

In order to represent the direction of the current along wire as it sweeps across the arc, a polar graph is traced by the tip of the vector at standard position. As it moves, there is a tangent vector. From the polar vector function, **r**(θ), **l**(θ) can be derived. As the magnitude is provided by "ds", only the direction is "extracted" from **r**(θ).

$$\hat{l}(\theta) = \frac{1}{\left|\vec{r'(\theta)}\right|} r'(\theta)$$

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$$\tau=\int_0^{2\pi}\vec{r}(\theta)\times(I\hat{l}(\theta)\times\vec{B})\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Let us find the derivative of **r** with respect to θ:

$$\vec{r}\,'(\theta)=\left(r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)\right)\vec{i}_\theta+\left(r'(\theta)\sin(\theta)+r(\theta)\cos(\theta)\right)\vec{j}_\theta$$

Using a change of basis/geometric argument as the basis are orthogonal unit vectors:

The magnitude is calculated as the root of the sum square of the components on said plane. There is a lovely lemma in the addendum for those who are curious.

$$|\vec{r}\,'(\theta)|=\sqrt{\left(r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)\right)^2+\left(r'(\theta)\sin(\theta)+r(\theta)\cos(\theta)\right)^2}$$

$$\begin{aligned}|\vec{r}\,'(\theta)^2|&=r'(\theta)^2\cos^2(\theta)-2r(\theta)r'(\theta)\sin(\theta)\cos(\theta)+r(\theta)^2\sin^2(\theta)\\&\quad+r'(\theta)^2\sin^2(\theta)+2r(\theta)r'(\theta)\sin(\theta)\cos(\theta)+r(\theta)^2\cos^2(\theta)\end{aligned}$$

$$\begin{aligned}&=r'(\theta)^2\cos^2(\theta)+r(\theta)^2\sin^2(\theta)\\&+r'(\theta)^2sin^2(\theta)+r(\theta)^2cos^2(\theta)\end{aligned}$$

$$\begin{aligned}&=r'(\theta)^2\cos^2(\theta)+r'(\theta)^2\sin^2(\theta)\\&+r(\theta)^2\cos^2(\theta)+r(\theta)^2\sin^2(\theta)\end{aligned}$$

$$=r'(\theta)^2+r(\theta)^2$$

Therefore:

$$|\vec{r}\,'(\theta)|=\sqrt{r'(\theta)^2+r(\theta)^2}=\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}$$

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Elimination of the Pesky Term

We will leverage the this property of the current direction vector.

$$\hat{l}(\theta)=\frac{1}{\left|\vec{r'(\theta)}\right|}r'(\theta)$$

$$\hat{l}(\theta)=\frac{1}{\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}}r'(\theta)$$

$$\tau=\int_{=0}^{2\pi}\vec{r}(\theta)\times(I\hat{l}(\theta)\times\vec{B})\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_{0}^{2\pi}\vec{r}(\theta)\times(I\frac{1}{\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}}r'(\theta)\times\vec{B})\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Scalars are the planewalkers of the cross product operation.

$$=\int_{0}^{2\pi}\vec{r}(\theta)\times(Ir'(\theta)\times\vec{B})\frac{1}{\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}}\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$\tau=\int_{0}^{2\pi}\vec{r}(\theta)\times(I\hat{l}(\theta)\times\vec{B})\sqrt{r(\theta)^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_{0}^{2\pi}\vec{r}(\theta)\times(I\vec{r'}(\theta)\times\vec{B})\,d\theta$$

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Triple Vector Product Expansion

$$\int_0^{2\pi} \vec{r}(\theta)\times(Ir'(\theta)\times\vec{B})\,d\theta = I\int_0^{2\pi} \vec{r}(\theta)\times(\vec{r}\,'(\theta)\times\vec{B})\,d\theta$$

There exists a theorem by the same name of the "Lagrange Formula" (he was a pretty smart guy):

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$I\int_0^{2\pi} \vec{r}(\theta)\times(\vec{r}\,'(\theta)\times\vec{B})\,d\theta = I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{B})\vec{r}\,'(\theta) - (\vec{r}(\theta)\cdot\vec{r}\,'(\theta))\vec{B}\,d\theta$$

$$= I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{B})\vec{r}\,'(\theta)\,d\theta - I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{r}\,'(\theta))\vec{B}\,d\theta$$

We shall deal with each term individually.

The Easy Part

$$-I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{r}\,'(\theta))\vec{B}\,d\theta \;= -I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{r}\,'(\theta))\,d\theta\,\vec{B}$$

Consider:

$$\int_0^{2\pi} \vec{r}(\theta)\cdot\vec{r}\,'(\theta)\,d\theta$$

Let us simplify that which is being integrated.

Remember:

$$\vec{r}(\theta) = r(\theta)\cos(\theta)\vec{i}_\theta + r(\theta)\sin(\theta)\vec{j}_\theta$$

$$\vec{r}\,'(\theta) = \left(r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)\right)\vec{i}_\theta + \left(r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)\right)\vec{j}_\theta$$

Therefore:

$$\vec{r}(\theta)\cdot\vec{r}\,'(\theta) = \left(r(\theta)\cos(\theta)\right)\left(r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)\right) + \left(r(\theta)\sin(\theta)\right)\left(r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)\right) + 0$$

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$$=r'(\theta)\cos(\theta)r(\theta)\cos(\theta)-r(\theta)\sin(\theta)r(\theta)\cos(\theta)+\left(r'(\theta)\sin(\theta)r(\theta)\sin(\theta)+r(\theta)\cos(\theta)r(\theta)\sin(\theta)\right)$$

$$=r'(\theta)r(\theta)\cos^2(\theta)+r'(\theta)r(\theta)\sin^2(\theta)-r(\theta)^2\sin(\theta)\cos(\theta)+r(\theta)^2\sin(\theta)\cos(\theta)$$
$$=r'(\theta)r(\theta)\cos^2(\theta)+r'(\theta)r(\theta)\sin^2(\theta)$$

$$=r'(\theta)r(\theta)$$

This leads to:

$$-I\int_0^{2\pi}(\vec{r}(\theta)\cdot\vec{r}'(\theta))\vec{B}\,d\theta\ =-I\int_0^{2\pi}r'(\theta)r(\theta)\,d\theta\,\vec{B}$$

However, this satisfies the change of variable theorem.

$$-I\int_0^{2\pi}r(\theta)\frac{dr}{d\theta}\,d\theta\,\vec{B}=-I\int_{r(0)}^{r(2\pi)}r(\theta)\,dr\,\vec{B}$$

Note: $r(2\pi)=r(0)=r_0$

$$=-I\int_{r_0}^{r_0}r(\theta)\,dr\,\vec{B}=0$$

As much as this appears suspect, the change of variable theorem does not require for the derivative to be added to the function, it's just that the bounds are the same. More on this "Strange Cancellation Theorem" is discussed in the Addendum.

The Less than Easy Part

As the second half of this equation is equal to zero, we return to a single integral.

$$\tau=I\int_0^{2\pi}(\vec{r}(\theta)\cdot\vec{B})\vec{r}'(\theta)\,d\theta-I\int_0^{2\pi}(\vec{r}(\theta)\cdot\vec{r}'(\theta))\vec{B}\,d\theta=I\int_0^{2\pi}(\vec{r}(\theta)\cdot\vec{B})\vec{r}'(\theta)\,d\theta-0$$

$$=I\int_0^{2\pi}(\vec{r}(\theta)\cdot\vec{B})\vec{r}'(\theta)\,d\theta$$

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Important Formulae

Before we proceed, there are two statements that we must keep an eye out for: the area of a polar function, and the cross product of the normal vector and the magnetic field.

The area formula:

$$A=\int_0^{2\pi}\frac{r(\theta)^2}{2}ds$$

The derivation of the cross product between the normal vector and the magnetic field.

$$\vec{B}=B_x\vec{i}_\theta+B_y\vec{j}_\theta+B_z\vec{k}_\theta$$

$$\hat{n}=\frac{1}{|\vec{r}(\theta)|}\vec{r}(\theta)\times\frac{1}{|\hat{r}'(\theta)|}\hat{r}'(\theta)$$

Where: I swear I'm not egotistic,

$$\vec{r}(\theta)=r(\theta)\cos(\theta)\vec{i}_\theta+r(\theta)\sin(\theta)\vec{j}_\theta$$

$$\hat{r}(\theta)=\cos(\theta)\vec{i}_\theta+\sin(\theta)\vec{j}_\theta$$

Leading to:

$$\hat{n}=\frac{1}{r(\theta)}\left(r(\theta)\cos(\theta)\vec{i}_\theta+r(\theta)\sin(\theta)\vec{j}_\theta\right)\times\frac{1}{1}\left(-sin(\theta)\vec{i}_\theta+cos(\theta)\vec{j}_\theta\right)$$

$$=\left(cos(\theta)\vec{i}_\theta+sin(\theta)\vec{j}_\theta\right)\times\left(-sin(\theta)\vec{i}_\theta+cos(\theta)\vec{j}_\theta\right)\{\{-1\ ^\wedge\ 2+\{1\ \text{over}\ 2\}\}1\ ^\wedge\ 4\}$$

Pseudodeterminant! (mneomic abuse of notation)

$$\hat{n}=\begin{vmatrix}\vec{i}_\theta & \vec{j}_\theta & \vec{k}_\theta\\ \cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\end{vmatrix}=0\ +\ 0\ +\ \left(\cos^2(\theta)+\sin^2(\theta)\right)\vec{k}_\theta\ =\ \vec{k}_\theta$$

Finally:

$$\vec{n}\times\vec{B}=\ \left(\vec{k}_\theta\right)\times\left(B_x\vec{i}_\theta+B_y\vec{j}_\theta+B_z\vec{k}_\theta\right)=-B_y\vec{i}_\theta+B_x\vec{j}_\theta$$

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The motivation behind this is that the magnetic dipole moment torque theorem the cross product has the area and the cross product of the normal and magnetic vectors.

$\tau = \vec{u} \times \vec{B}$ where $\vec{u} = IA\,\hat{n}$.

$$\tau = IA\,\hat{n} \times \vec{B} = I \int_0^{2\pi} \frac{r(\theta)^2}{2}\, ds\, \hat{n} \times \vec{B} = I \int_0^{2\pi} \frac{r(\theta)^2}{2}\, ds \left(-B_y \vec{i_\theta} + B_x \vec{j_\theta}\right)$$

Finally, because eventually, we shall be proving component by component:

$$\tau = -I \int_0^{2\pi} \frac{r(\theta)^2}{2}\, ds\, B_y \vec{i_\theta} + I \int_0^{2\pi} \frac{r(\theta)^2}{2}\, ds\, B_x \vec{j_\theta}$$

Note that the magnetic y component (in the new coordinate system) is associated with i and the x with j.

Birthing the Behemoth Integral

$$I \int_0^{2\pi} (\vec{r}(\theta) \cdot \vec{B}) \vec{r}\,'(\theta)\, d\theta$$

Consider the term being integrated:

$$(\vec{r}(\theta) \cdot \vec{B}) \vec{r}\,'(\theta)$$

Let us expand the dot product (focusing upon it):

$$(\vec{r}(\theta) \cdot \vec{B}) = \left(\cos(\theta)\vec{i_\theta} + \sin(\theta)\vec{j_\theta}\right) \cdot \left(B_x \vec{i_\theta} + B_y \vec{j_\theta} + B_z \vec{k_\theta}\right)$$

$$\left(B_x r(\theta)\cos(\theta) + B_y r(\theta)\sin(\theta)\right)$$

And we shall distribute it into the original vector.

$$\left(B_x r(\theta)\cos(\theta) + B_y r(\theta)\sin(\theta)\right)\left(r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)\right)\vec{i_\theta} + \left(r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)\right)\vec{j_\theta}$$

$$= \left(B_x r(\theta)\cos(\theta)\right)\left(r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)\right)\vec{i_\theta} + \left(r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)\right)\vec{j_\theta}$$
$$+ \left(B_y r(\theta)\sin(\theta)\right)\left(r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)\right)\vec{i_\theta} + \left(r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)\right)\vec{j_\theta}$$

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$$
\begin{aligned}
&=\left(B_x r(\theta)\cos(\theta)r'(\theta)\cos(\theta)-B_x r(\theta)\cos(\theta)r(\theta)\sin(\theta)\right)\vec{i}_\theta\\
&+\left(B_x r(\theta)\cos(\theta)r'(\theta)\sin(\theta)+B_x r(\theta)\cos(\theta)r(\theta)\cos(\theta)\right)\vec{j}_\theta\\
&+\left(B_y r(\theta)\sin(\theta)r'(\theta)\cos(\theta)-B_y r(\theta)\sin(\theta)r(\theta)\sin(\theta)\right)\vec{i}_\theta\\
&+\left(B_y r(\theta)\sin(\theta)r'(\theta)\sin(\theta)+B_y r(\theta)\sin(\theta)r(\theta)\cos(\theta)\right)\vec{j}_\theta
\end{aligned}
$$

$$
\begin{aligned}
&=\left(B_x r(\theta)\cos(\theta)r'(\theta)\cos(\theta)-B_x r(\theta)\cos(\theta)r(\theta)\sin(\theta)\right)\vec{i}_\theta\\
&+\left(B_y r(\theta)\sin(\theta)r'(\theta)\cos(\theta)-B_y r(\theta)\sin(\theta)r(\theta)\sin(\theta)\right)\vec{i}_\theta\\
&+\left(B_x r(\theta)\cos(\theta)r'(\theta)\sin(\theta)+B_x r(\theta)\cos(\theta)r(\theta)\cos(\theta)\right)\vec{j}_\theta\\
&+\left(B_y r(\theta)\sin(\theta)r'(\theta)\sin(\theta)+B_y r(\theta)\sin(\theta)r(\theta)\cos(\theta)\right)\vec{j}_\theta
\end{aligned}
$$

$$
\begin{aligned}
&=\left(B_x\cos^2(\theta)r(\theta)r'(\theta)-B_x r(\theta)^2\cos(\theta)\sin(\theta)\right)\vec{i}_\theta\\
&+\left(B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)-B_y r(\theta)^2\sin^2(\theta)\right)\vec{i}_\theta\\
&+\left(B_x\cos(\theta)\sin(\theta)r(\theta)r'(\theta)+B_x r(\theta)^2\cos^2(\theta)\right)\vec{j}_\theta\\
&+\left(B_y\sin^2(\theta)r(\theta)r'(\theta)+B_y r(\theta)^2\sin(\theta)\cos(\theta)\right)\vec{j}_\theta
\end{aligned}
$$

$$
\begin{aligned}
&=B_x\cos^2(\theta)r(\theta)r'(\theta)\vec{i}_\theta-B_x r(\theta)^2\cos(\theta)\sin(\theta)\vec{i}_\theta\\
&+B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)\vec{i}_\theta-B_y r(\theta)^2\sin^2(\theta)\vec{i}_\theta\\
&+B_x\cos(\theta)\sin(\theta)r(\theta)r'(\theta)\vec{j}_\theta+B_x r(\theta)^2\cos^2(\theta)\vec{j}_\theta\\
&+B_y\sin^2(\theta)r(\theta)r'(\theta)\vec{j}_\theta+B_y r(\theta)^2\sin(\theta)\cos(\theta)\vec{j}_\theta
\end{aligned}
$$

Finally, this may seem frightening, but a significant amount of terms cancel out as shown in the later sections.

$$I\int_0^{2\pi}(\vec{r}(\theta)\cdot\vec{B})\vec{r}'(\theta)\,d\theta=$$

$$
\begin{aligned}
&I\int_0^{2\pi}B_x\cos^2(\theta)r(\theta)r'(\theta)\,d\theta\,\vec{i}_\theta-I\int_0^{2\pi}B_x r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta\\
&+I\int_0^{2\pi}B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)\,d\theta\,\vec{i}_\theta-I\int_0^{2\pi}B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta\\
&+I\int_0^{2\pi}B_x\cos(\theta)\sin(\theta)r(\theta)r'(\theta)\,d\theta\,\vec{j}_\theta+I\int_0^{2\pi}B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta\\
&+I\int_0^{2\pi}B_y\sin^2(\theta)r(\theta)r'(\theta)\,d\theta\,\vec{j}_\theta+I\int_0^{2\pi}B_y r(\theta)^2\sin(\theta)\cos(\theta)\,d\theta\,\vec{j}_\theta
\end{aligned}
$$

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Canceling i terms

Let us focus upon the terms that are along the i theta direction.

$$I\int_0^{2\pi} B_x\cos^2(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta - I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)d\theta\vec{i}_\theta$$
$$+I\int_0^{2\pi} B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)d\theta\vec{i}_\theta$$

$$=I\int_0^{2\pi} B_x\cos^2(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta + I\int_0^{2\pi} B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta$$
$$-I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)d\theta\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)d\theta\vec{i}_\theta$$

Two of the terms are multiplied by r and its integral. Thus, the Strange Cancellation Theorem can be utilized!

Note, theta can be a piecewise function of of r, and the integral can be broken up so the; however, **since the interval is literally from $\mathbf{r_0}$ to $\mathbf{r_0}$**. Whichever function is used for the interval that contains $\mathbf{r_0}$ **t**hat provides the proper value **ultimately matter because the integral is zero regardless due to the interval itself.** This function only needs to be well defined for $\mathbf{r_0}$, but it can be for all relevant θ.

$$I\int_0^{2\pi} B_x\cos^2(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta = I\int_{r(0)}^{r(2\pi)} B_x\cos^2(\theta)rdr\,\vec{i}_\theta = I\int_{r_0}^{r_0} B_x\cos^2(\theta)rdr\,\vec{i}_\theta = 0$$

$$I\int_0^{2\pi} B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta = I\int_{r(0)}^{r(2\pi)} B_y\sin(\theta)\cos(\theta)rdr\,\vec{i}_\theta = I\int_{r_0}^{r_0} B_y\sin(\theta)\cos(\theta)rdr\,\vec{i}_\theta = 0$$

$$I\int_0^{2\pi} B_x\cos^2(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta + I\int_0^{2\pi} B_y\sin(\theta)\cos(\theta)r(\theta)r'(\theta)d\theta\vec{i}_\theta$$
$$-I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)d\theta\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)d\theta\vec{i}_\theta$$
$$=0+0-I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)d\theta\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)d\theta\vec{i}_\theta$$

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$$=-I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta$$

There is another term that cancels out, but through different means: integration by parts.

$$-I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta = -I\,B_x\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta$$

Consider the inner integral.

$$\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta$$

$$u=r(\theta)^2,\ \frac{dv}{d\theta}=\cos(\theta)\sin(\theta)$$

$$\frac{du}{d\theta}=2r(\theta)\frac{dr}{d\theta},\quad v=\frac{\sin^2(\theta)}{2}$$

$$\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta = r(\theta)^2\frac{\sin^2(\theta)}{2}\Big|_0^{2\pi} + \int_0^{2\pi}\sin^2(\theta)r(\theta)\frac{dr}{d\theta}\,d\theta$$

$$\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta = 0+0=0$$

At the left term sine at 0 and 2π is 0 meaning it equals 0 while the rig I swear I'm not egotistic,ht term becomes 0 due to the Strange Cancellation Theorem.

$$=-I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta$$

$$=0-I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta \qquad \text{B_x}$$

$$=-I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta$$

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The i term that becomes the good part

$$-I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta=-I\,B_y\int_0^{2\pi} r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta$$

Let us focus upon the integral.

$$\int_0^{2\pi} r(\theta)^2\sin^2(\theta)\,d\theta$$

$$u=\sin^2(\theta),\frac{dv}{d\theta}=r(\theta)^2$$

$$\frac{du}{d\theta}=2\sin(\theta)\cos(\theta),\quad v=\int r(\theta)^2\,d\theta=F(\theta)$$

$$\int_0^{2\pi} r(\theta)^2\sin^2(\theta)\,d\theta=\sin^2(\theta)\int r(\theta)^2\,d\theta\Big|_0^{2\pi}-\int_0^{2\pi}2\sin(\theta)\cos(\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta$$

$$=0-\int_0^{2\pi}2\sin(\theta)\cos(\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta$$

The following emerges from the sine angle sum theorem:

$$\sin(\theta)\cos(\theta)=\frac{\sin(2\theta)}{2}$$

$$-\int_0^{2\pi}2\sin(\theta)\cos(\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta=-\int_0^{2\pi}\sin(2\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta$$

$$u=\int r(\theta)^2\,d\theta,\frac{dv}{d\theta}=\sin(2\theta)$$

$$\frac{du}{d\theta}=r(\theta)^2\,d\theta,\quad v=\frac{-\cos(2\theta)}{2}$$

I swear I'm not egotistic,

$$-\int_0^{2\pi}\sin(2\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta=-\left[\frac{-\cos(2\theta)}{2}\int r(\theta)^2\,d\theta\Big|_0^{2\pi}-\int_0^{2\pi}-\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta\right]$$

$$=\frac{\cos(2\theta)}{2}\int r(\theta)^2\,d\theta\Big|_0^{2\pi}-\int_0^{2\pi}\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta$$

Consider the 2nd term:

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$$-\int_0^{2\pi} \frac{\cos(2\theta)}{2} r(\theta)^2 d\theta$$

$$u = r(\theta)^2, \frac{dv}{d\theta} = \frac{\cos(2\theta)}{2}$$

$$\frac{du}{d\theta} = 2r(\theta)\frac{dr}{d\theta}, \quad v = \frac{\sin(2\theta)}{4}$$

$$-\int_0^{2\pi} \frac{\cos(2\theta)}{2} r(\theta)^2 d\theta = -\left[\frac{\sin(2\theta)}{4} r(\theta)^2 \Big|_0^{2\pi} - \int_0^{2\pi} \frac{\sin(2\theta)}{4} 2r(\theta)\frac{dr}{d\theta} d\theta \right]$$

The first term becomes zero, because of sine equaling zero regardless of the value of r (which is define). The 2nd term will become zero because of the Strange Cancellation Theorem (this is where we begin leaving it as an "exercise to the reader" or via "proof by obviousness".

$$-\int_0^{2\pi} \frac{\cos(2\theta)}{2} r(\theta)^2 d\theta = -(0-0) = 0$$

Consider the first term:

$$= \frac{\cos(2\theta)}{2} \int r(\theta)^2 d\theta \Big|_0^{2\pi} = \frac{\cos(2\theta)}{2} F(\theta) \Big|_0^{2\pi} = \left(F(2\pi)\frac{\cos(4\pi)}{2} \right) - \left(F(0)\frac{\cos(0)}{2} \right)$$

$$= \left(F(2\pi)\frac{1}{2} \right) - \left(F(0)\frac{1}{2} \right) = \frac{1}{2}(F(2\pi) - F(0))$$

By invoking the Fundamental Theorem of Calculus, we reveal the area of a polar function:

$$\frac{1}{2}(F(2\pi) - F(0)) = \frac{1}{2}\int_0^{2\pi} r(\theta)^2 d\theta = \int_0^{2\pi} \frac{1}{2} r(\theta)^2 d\theta$$

Therefore

$$\frac{\cos(2\theta)}{2} \int r(\theta)^2 d\theta \Big|_0^{2\pi} - \int_0^{2\pi} \frac{\cos(2\theta)}{2} r(\theta)^2 d\theta = \int_0^{2\pi} \frac{1}{2} r(\theta)^2 d\theta$$

And by extension (remember this was the integral that triggered the chain of integration by parts:

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$$\int_0^{2\pi} r(\theta)^2 \sin^2(\theta)\, d\theta = \int_0^{2\pi} \frac{1}{2} r(\theta)^2\, d\theta$$

And by further extension:

$$I \int_0^{2\pi} B_x r(\theta)^2 \cos^2(\theta)\, d\theta\, \vec{j}_\theta = I B_x \int_0^{2\pi} r(\theta)^2 \cos^2(\theta)\, d\theta\, \vec{j}_\theta$$

$$-I \int_0^{2\pi} B_y r(\theta)^2 \sin^2(\theta)\, d\theta\, \vec{i}_\theta = -I B_y \int_0^{2\pi} r(\theta)^2 \sin^2(\theta)\, d\theta\, \vec{i}_\theta$$

$$= -I B_y \int_0^{2\pi} \frac{1}{2} r(\theta)^2\, d\theta\, \vec{i}_\theta$$

WHOAHHHH!!! This proves the i term.

Debriefing for the j terms.

The procedure for the j terms will be similar. Shall we leave it as an exercise to the reader? Nay. But any invocation of the Strange Cancellation Theorem shall be, for the result is **zero** and the change of variable that sets it up is *obvious*.

$$I \int_0^{2\pi} B_x \cos(\theta) \sin(\theta) r(\theta) r'(\theta)\, d\theta\, \vec{j}_\theta + I \int_0^{2\pi} B_x r(\theta)^2 \cos^2(\theta)\, d\theta\, \vec{j}_\theta$$

$$+ I \int_0^{2\pi} B_y \sin^2(\theta) r(\theta) r'(\theta)\, d\theta\, \vec{j}_\theta + I \int_0^{2\pi} B_y r(\theta)^2 \sin(\theta) \cos(\theta)\, d\theta\, \vec{j}_\theta$$

Canceling j terms

We recognize the Strange Cancellation Theorem and move on thanks to the **Obvious Theorem** (just kidding). Any concerns with converting theta into a function of r, is mentioned in the "Canceling i terms" section. (I would not recommend skipping around the proof if this is a concern that cannot be resolved.)

$$I \int_0^{2\pi} B_x \cos(\theta) \sin(\theta) r(\theta) r'(\theta)\, d\theta\, \vec{j}_\theta = 0$$

$$+ I \int_0^{2\pi} B_y \sin^2(\theta) r(\theta) r'(\theta)\, d\theta\, \vec{j}_\theta = 0$$

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$$I\int_0^{2\pi} B_y r(\theta)^2\sin(\theta)\cos(\theta)\,d\theta\,\vec{j}_\theta = I\,B_y\int_0^{2\pi} r(\theta)^2\sin(\theta)\cos(\theta)\,d\theta\,\vec{j}_\theta$$

Consider the inner integral (this was graciously copied from the i section.)

$$\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta$$

$u = r(\theta)^2, \frac{dv}{d\theta} = \cos(\theta)\sin(\theta)$ The Less than Easy Part

$$\frac{du}{d\theta} = 2r(\theta)\frac{dr}{d\theta},\quad v = \frac{\sin^2(\theta)}{2}$$

$$\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta = r(\theta)^2\frac{\sin^2(\theta)}{2}\Big|_0^{2\pi} + \int_0^{2\pi}\sin^2(\theta)r(\theta)\frac{dr}{d\theta}\,d\theta$$

$$\int_0^{2\pi} r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta = 0+0=0$$

$$I\,B_y\int_0^{2\pi} r(\theta)^2\sin(\theta)\cos(\theta)\,d\theta\,\vec{j}_\theta = I\,B_y = 0$$ The j terms that become the good part

Therefore, like the Hunger Games:Prelude

$$I\int_0^{2\pi} B_x\cos(\theta)\sin(\theta)r(\theta)r'(\theta)\,d\theta\,\vec{j}_\theta + I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta$$
$$+I\int_0^{2\pi} B_y\sin^2(\theta)r(\theta)r'(\theta)\,d\theta\,\vec{j}_\theta + I\int_0^{2\pi} B_y r(\theta)^2\sin(\theta)\cos(\theta)\,d\theta\,\vec{j}_\theta$$

$$=0+I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta+0+0=I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta$$

Survival of the fittest - and fattest - integrals.

$$I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta = I\,B_x\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta$$

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The j term that become the good part

$$I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta = I\,B_x\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta$$

Consider the inner integral:

$$\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta$$

$$u=\cos^2(\theta),\ \frac{dv}{d\theta}=r(\theta)^2$$

$$\frac{du}{d\theta}=-2\sin(\theta)\cos(\theta),\quad v=\int r(\theta)^2\,d\theta=F(\theta)$$

$$\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta=\frac{\sin(2\theta)}{4}r(\theta)^2\Big|_0^{2\pi}-\int_0^{2\pi}\frac{\sin(2\theta)}{4}2r(\theta)\frac{dr}{d\theta}\,d\theta$$

$$\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta=\cos^2(\theta)\int r(\theta)^2\,d\theta\Big|_0^{2\pi}-\int_0^{2\pi}-2\sin(\theta)\cos(\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta$$

$$=\cos^2(\theta)\int r(\theta)^2\,d\theta\Big|_0^{2\pi}+\int_0^{2\pi}2\sin(\theta)\cos(\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta$$

Consider:

$$\cos^2(\theta)\int r(\theta)^2\,d\theta\Big|_0^{2\pi}=\cos^2(\theta)F(\theta)\Big|_0^{2\pi}=\big(\cos^2(2\pi)F(2\pi)\big)-\big(\cos^2(0)F(0)\big)$$

$$=\big(1\cdot F(2\pi)\big)-\big(1\cdot F(0)\big)=\int_0^{2\pi} r(\theta)^2\,d\theta$$

We hope other term eventually becomes $-\frac{1}{2}\int_0^{2\pi} r(\theta)^2\,d\theta$

Consider:

$$\int_0^{2\pi}2\sin(\theta)\cos(\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta=\int_0^{2\pi}\sin(2\theta)\Big[\int r(\theta)^2\,d\theta\Big]d\theta$$

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$$u=\int r(\theta)^2\,d\theta,\frac{dv}{d\theta}=\sin(2\theta)$$
$$\frac{du}{d\theta}=r(\theta)^2\,d\theta,\quad v=\frac{-\cos(2\theta)}{2}$$
$$\int_0^{2\pi}\sin(2\theta)\left[\int r(\theta)^2\,d\theta\right]d\theta=\left[\frac{-\cos(2\theta)}{2}\int r(\theta)^2\,d\theta\right|_0^{2\pi}-\int_0^{2\pi}-\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta\Bigg]$$
$$=-\frac{\cos(2\theta)}{2}\int r(\theta)^2\,d\theta\Bigg|_0^{2\pi}+\int_0^{2\pi}\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta$$

Consider the 2nd term:

$$\int_0^{2\pi}\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta$$
$$u=r(\theta)^2,\frac{dv}{d\theta}=\frac{\cos(2\theta)}{2}$$
$$\frac{du}{d\theta}=2\mathrm{r}(\theta)\frac{dr}{d\theta},\quad v=\frac{sin(2\theta)}{4}$$
$$\int_0^{2\pi}\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta=\frac{\sin(2\theta)}{4}r(\theta)^2\Bigg|_0^{2\pi}-\int_0^{2\pi}\frac{\sin(2\theta)}{4}2r(\theta)\frac{dr}{d\theta}\,d\theta$$

The first term becomes zero, because of sine equaling zero regardless of the value of r (which is define). The 2nd term will become zero because of the Strange Cancellation Theorem (this is where we begin leaving it as an "exercise to the reader" or via "proof by obviousness".

$$\int_0^{2\pi}\frac{\cos(2\theta)}{2}r(\theta)^2\,d\theta=-(0-0)=0$$

Consider the first term:

$$=-\frac{\cos(2\theta)}{2}\int r(\theta)^2\,d\theta\Bigg|_0^{2\pi}=-\frac{\cos(2\theta)}{2}F(\theta)\Bigg|_0^{2\pi}=-\left[\left(F(2\pi)\frac{\cos(4\pi)}{2}\right)-\left(F(0)\frac{\cos(0)}{2}\right)\right]$$

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$$=-\left[\left(F(2\pi)\frac{1}{2}\right)-\left(F(0)\frac{1}{2}\right)\right]=-\frac{1}{2}(F(2\pi)-F(0))$$

By invoking the Fundamental Theorem of Calculus, we reveal the area of a polar function:

$$-\frac{1}{2}(F(2\pi)-F(0))=-\frac{1}{2}\int_0^{2\pi} r(\theta)^2\,d\theta=-\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta$$

Our thoughts and prayers worked (for once).

$$\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta=\cos^2(\theta)\int r(\theta)^2\,d\theta\Big|_0^{2\pi}+\int_0^{2\pi}2\sin(\theta)\cos(\theta)\left[\int r(\theta)^2\,d\theta\right]d\theta$$

$$=\int_0^{2\pi} r(\theta)^2\,d\theta=-\frac{1}{2}\int_0^{2\pi} r(\theta)^2\,d\theta=\frac{1}{2}\int_0^{2\pi} r(\theta)^2\,d\theta=\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta$$

Therefore by extension:

$$I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta=I\,B_x\int_0^{2\pi} r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta=I\,B_x\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta$$

Putting it all together

In the almost last 10 pages, these results were proven.

$$-I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta=-I\,B_y\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta\,\vec{i}_\theta$$

$$I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta=I\,B_x\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta$$

From the behemoth integral, we now know it is not so frightening after all.

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$$\tau = I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{B})\vec{r}\,'(\theta)\,d\theta =$$

$$I\int_0^{2\pi} B_x\cos^2(\theta)r(\theta)r\,'(\theta)\,d\theta\,\vec{i}_\theta - I\int_0^{2\pi} B_x r(\theta)^2\cos(\theta)\sin(\theta)\,d\theta\,\vec{i}_\theta$$

$$+I\int_0^{2\pi} B_y\sin(\theta)\cos(\theta)r(\theta)r\,'(\theta)\,d\theta\,\vec{i}_\theta - I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta$$

$$+I\int_0^{2\pi} B_x\cos(\theta)\sin(\theta)r(\theta)r\,'(\theta)\,d\theta\,\vec{j}_\theta + I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta$$

$$+I\int_0^{2\pi} B_y\sin^2(\theta)r(\theta)r\,'(\theta)\,d\theta\,\vec{j}_\theta + I\int_0^{2\pi} B_y r(\theta)^2\sin(\theta)\cos(\theta)\,d\theta\,\vec{j}_\theta$$

$$\tau = I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{B})\vec{r}\,'(\theta)\,d\theta = -I\int_0^{2\pi} B_y r(\theta)^2\sin^2(\theta)\,d\theta\,\vec{i}_\theta + I\int_0^{2\pi} B_x r(\theta)^2\cos^2(\theta)\,d\theta\,\vec{j}_\theta$$

$$\tau = I\int_0^{2\pi} (\vec{r}(\theta)\cdot\vec{B})\vec{r}\,'(\theta)\,d\theta = -I\,B_y\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta\,\vec{i}_\theta + I\,B_x\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta$$

Remember:

$\tau = \vec{u}\times\vec{B}$
$\vec{u} = IA\,\hat{n}$
$\tau = IA\,\hat{n}\times\vec{B}$

$$\vec{n}\times\vec{B} = \left(\vec{k}_\theta\right)\times\left(B_x\vec{i}_\theta + B_y\vec{j}_\theta + B_z\vec{k}_\theta\right) = -B_y\vec{i}_\theta + B_x\vec{j}_\theta$$

$$\tau = \vec{u}\times\vec{B} = IA\,\hat{n}\times\vec{B} = IA\left(-B_y\vec{i}_\theta + B_x\vec{j}_\theta\right)$$

'

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Let us Factor:

$$\tau = I\int_0^{2\pi}\frac{1}{2}r(\theta)^2\,d\theta\left(-B_y\vec{i}_\theta + B_x\right)$$
$$= IA\left(-B_y\vec{i}_\theta + B_x\right)$$
$$= IA\,\hat{n}\times\vec{B}$$
$$\tau = \vec{u}\times\vec{B}$$

It is proven Q.E.D.

☆ **"It is golden"**

(The pentagram has golden ratios in it)

Equal Everywhere Magnetic Dipole Torque Theorem

"The torque for a loop of wire is equal at all points on the plane that it encloses."

$$\tau = IA\,\hat{n}\times\vec{B}$$

Notice that the torque only relies upon area. Which means that if another polar function which produces the same set of points with the same orientation, it will have the same area and the same normal vector produced. Since the only thing that is changed is the origin of the polar function, the current is the same as well.

Conclusion

Sometimes proofs are not elegant. In mathematics, breaking big functions into easier pieces often helps a lot. Sometimes things become complicated, yet without that complexity, we would not be able to simplify certain things. Textbooks can wave their hand for the sake of the student, but sometimes all immediate resources will do so as well and it is up to the student to seek that truth on their own.

But it is in that quest that one gains deeper insights and understanding.

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Addendum

Magnitude of a vector expressed as a sum of orthogonal unit vectors.

Where $\vec{r}=r_x\hat{u}+r_y\hat{v}+r_z\hat{w}$

$|\hat{u}|=|\hat{v}|\hat{=}1$

$\hat{w}=\hat{u}\times\hat{v}$

$\hat{u}\perp\hat{v}$

Then: $|\vec{r}|=\sqrt{r_x^2+r_y^2+r_z^2}$

The sum of squares of the components whose unit vectors are 1. As they are orthogonal, their cross product must equal zero.

$\hat{u}=\langle u_x,\mathrm{u}_y,\mathrm{u}_z\rangle,\ \ \hat{v}=\langle v_x,v_y,v_z\rangle,\ \ \hat{w}=\langle w_x,w_y,w_z\rangle$

$\hat{u}\perp\hat{v}$

$\hat{w}=\hat{u}\times\hat{v}$

$|\hat{u}|=|\hat{v}|=|\hat{\hat{w}}|=1$

Let **r** be a vector expressed as a linear combination of the unit vectors.

$\tau=\vec{u}\times\vec{B}=IA\,\hat{n}\times\vec{B}=IA\left(-B_y\vec{i}_\theta+B_x\vec{j}_\theta\ \right)$

$\vec{r}=r_x\hat{u}+r_y\hat{v}+r_z\hat{w}=\langle r_xu_x,r_xu_y,r_xu_z\rangle+\langle r_yv_x,r_yv_y,r_yv_z\rangle+\langle r_zw_x,r_zw_y,r_zw_z\rangle$
$=\langle r_xu_x+r_yv_x+r_zw_x,r_xu_y+r_yv_y+r_zw_y,r_xu_z+r_yv_z+r_zw_z\rangle$

These lemmas will come in handy:

$\hat{u}\perp\hat{v},\hat{v}\perp\hat{w},\hat{u}\perp\hat{w}$

$|\hat{u}|=|\hat{v}|=|\hat{w}|=1$

$u_x^2+u_y^2+u_z^2=v_x^2+v_y^2+v_z^2=w_x^2+w_y^2+w_z^2=1$

$\hat{u}\cdot\hat{v}=0,\hat{v}\cdot\hat{w}=0,\hat{u}\cdot\hat{w}=0$

$u_x^2+u_y^2+u_z^2=v_x^2+v_y^2+v_z^2=w_x^2+w_y^2+w_z^2=1$

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Magnetic Dipole Torque Theorem Proof for Polar Curves

Raven Asher Raziel
12/10/02024

$$\hat{u}\cdot\hat{v}=\langle u_x,u_y,u_z\rangle\cdot\langle v_x,v_y,v_z\rangle=u_xv_x+u_yv_y+u_zv_z=0$$

$$\hat{v}\cdot\hat{w}=\langle v_x,v_y,v_z\rangle\cdot\langle w_x,w_y,w_z\rangle=v_xw_x+v_yw_y+v_zw_z=0$$

$$\hat{u}\cdot\hat{w}=\langle u_x,u_y,u_z\rangle\cdot\langle w_x,w_y,w_z\rangle=u_xw_x+u_yw_y+u_zw_z=0$$

$$\hat{w}=\hat{u}\times\hat{v}$$

$$\hat{w}=\langle u_yv_z-u_zv_y,u_zv_x-u_xv_z,\mathrm{u}_xv_y-u_yv_x\rangle$$

Back to the proof

$$\begin{aligned}\vec{r}&=r_x\hat{u}+r_y\hat{v}+r_z\hat{w}\\&=\langle r_xu_x,r_xu_y,r_xu_z\rangle\\&+\langle r_yv_x,r_yv_y,r_yv_z\rangle\\&+\langle r_zw_x,r_zw_y,r_zw_z\rangle\end{aligned}$$

$$¿=\langle r_xu_x+r_yv_x+r_zw_x,r_xu_y+r_yv_y+r_zw_y,r_xu_z+r_yv_z+r_zw_z\rangle$$

$$|\vec{r}|=\sqrt{\left(r_xu_x+r_yv_x+r_zw_x\right)^2+\left(r_xu_y+r_yv_y+r_zw_y\right)^2+\left(r_xu_z+r_yv_z+r_zw_z\right)^2}$$

$$|\vec{r}|^2=\left(r_xu_x+r_yv_x+r_zw_x\right)^2+\left(r_xu_y+r_yv_y+r_zw_y\right)^2+\left(r_xu_z+r_yv_z+r_zw_z\right)^2$$

$$\begin{aligned}|\vec{r}|^2=&\left(r_xu_x\left(r_xu_x+r_yv_x+r_zw_x\right)+r_yv_x\left(r_xu_x+r_yv_x+r_zw_x\right)+r_zw_x\left(r_xu_x+r_yv_x+r_zw_x\right)\right)\\&+\left(r_xu_y\left(r_xu_y+r_yv_y+r_zw_y\right)+r_yv_y\left(r_xu_y+r_yv_y+r_zw_y\right)+r_zw_y\left(r_xu_y+r_yv_y+r_zw_y\right)\right)\\&+\left(r_xu_z\left(r_xu_z+r_yv_z+r_zw_z\right)+r_yv_z\left(r_xu_z+r_yv_z+r_zw_z\right)+r_zw_z\left(r_xu_z+r_yv_z+r_zw_z\right)\right)\end{aligned}$$

$$\begin{aligned}|\vec{r}|^2=&\left(r_xu_xr_xu_x+r_xu_xr_yv_x+r_xu_xr_zw_x\right)+\left(r_yv_xr_xu_x+r_yv_xr_yv_x+r_yv_xr_zw_x\right)+\left(r_zw_xr_xu_x+r_zw_xr_yv_x+r_zw_xr_zw_x\right)\\&+\left(r_xu_yr_xu_y+r_xu_yr_yv_y+r_xu_yr_zw_y\right)+\left(r_yv_yr_xu_y+r_yv_yr_yv_y+r_yv_yr_zw_y\right)+\left(r_zw_yr_xu_y+r_zw_yr_yv_y+r_zw_yr_zw_y\right)\\&+\left(r_xu_zr_xu_z+r_xu_zr_yv_z+r_xu_zr_zw_z\right)+\left(r_yv_zr_xu_z+r_yv_zr_yv_z+r_yv_zr_zw_z\right)+\left(r_zw_zr_xu_z+r_zw_zr_yv_z+r_zw_zr_zw_z\right)\end{aligned}$$

$$\begin{aligned}|\vec{r}|^2=&\left(r_x^2u_x^2+r_xr_yu_xv_x+r_xr_zu_xw_x\right)+\left(r_xr_yu_xv_x+r_y^2v_x^2+r_yr_zv_xw_x\right)+\left(r_xr_zu_xw_x+r_yr_zv_xw_x+r_z^2w_x^2\right)\\&+\left(r_x^2u_y^2+r_xr_yu_yv_y+r_xr_zu_yw_y\right)+\left(r_xr_yu_yv_y+r_y^2v_y^2+r_yr_zv_yw_y\right)+\left(r_xr_zu_yw_y+r_yr_zv_yw_y+r_z^2w_y^2\right)\\&+\left(r_x^2u_z^2+r_xr_yu_zv_z+r_xr_zu_zw_z\right)+\left(r_xr_yu_zv_z+r_y^2v_z^2+r_yr_zv_zw_z\right)+\left(r_xr_zu_zw_z+r_yr_zv_zw_z+r_z^2w_z^2\right)\end{aligned}$$

Organize

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Magnetic Dipole Torque Theorem Proof for Polar Curves

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12/10/02024

$$|\vec{r}|^2=r_x^2u_x^2+r_xr_yu_xv_x+r_xr_zu_xw_x+r_xr_yu_xv_x+r_y^2v_x^2+r_yr_zv_xw_x+r_xr_zu_xw_x+r_yr_zv_xw_x+r_z^2w_x^2$$
$$+r_x^2u_y^2+r_xr_yu_yv_y+r_xr_zu_yw_y+r_xr_yu_yv_y+r_y^2v_y^2+r_yr_zv_yw_y+r_xr_zu_yw_y+r_yr_zv_yw_y+r_z^2w_y^2$$
$$+r_x^2u_z^2+r_xr_yu_zv_z+r_xr_zu_zw_z+r_xr_yu_zv_z+r_y^2v_z^2+r_yr_zv_zw_z+r_xr_zu_zw_z+r_yr_zv_zw_z+r_z^2w_z^2$$

Rearrange terms for factoring (what is about to happen will be quite beautiful.

$$|\vec{r}|^2=r_x^2u_x^2+r_x^2u_y^2+r_x^2u_z^2+r_xr_yu_xv_x+r_xr_yu_yv_y+r_xr_yu_zv_z+r_xr_zu_xw_x+r_xr_zu_yw_y+r_xr_zu_zw_z$$
$$+r_y^2v_x^2+r_y^2v_y^2+r_y^2v_z^2+r_xr_yu_xv_x+r_xr_yu_yv_y+r_xr_yu_zv_z+r_yr_zv_xw_x+r_yr_zv_yw_y+r_yr_zv_zw_z$$
$$+r_z^2w_x^2+r_z^2w_y^2+r_z^2w_z^2+r_xr_zu_xw_x+r_xr_zu_yw_y+r_xr_zu_zw_z+r_yr_zv_xw_x+r_yr_zv_yw_y+r_yr_zv_zw_z$$

$$|\vec{r}|^2=r_x^2\left(u_x^2+u_y^2+u_z^2\right)+r_xr_y\left(u_xv_x+u_yv_y+u_zv_z\right)+r_xr_z\left(u_xw_x+u_yw_y+u_zw_z\right)$$
$$+r_y^2\left(v_x^2+v_y^2+v_z^2\right)+r_xr_y\left(u_xv_x+u_yv_y+u_zv_z\right)+r_yr_z\left(v_xw_x+v_yw_y+v_zw_z\right)\quad\{\{-1\wedge 2+\{1\text{ over }2\}\}1\wedge 4\}$$
$$+r_z^2\left(w_x^2+w_y^2+w_z^2\right)+r_xr_z\left(u_xw_x+u_yw_y+u_zw_z\right)+r_yr_z\left(v_xw_x+v_yw_y+v_zw_z\right)$$

Remember that:

$$|\hat{u}|=|\hat{v}|=|\hat{w}|=1$$

WHOAH!!!!

$$\left(|\vec{r}|\right)^2=r_x^2\left(|\hat{u}|\right)+r_xr_y\left(\hat{u}\cdot\hat{v}\right)+r_xr_z\left(\hat{u}\cdot\hat{w}\right)$$
$$+r_y^2\left(|\hat{v}|\right)+r_xr_y\left(\hat{u}\cdot\hat{v}\right)+r_yr_z\left(\hat{v}\cdot\hat{w}\right)$$
$$+r_z^2\left(|\hat{w}|\right)+r_xr_z\left(\hat{u}\cdot\hat{w}\right)+r_yr_z\left(\hat{v}\cdot\hat{w}\right)$$

As **u,v, and w** are orthogonal unit vectors:

$$\left(|\vec{r}|\right)^2=r_x^2(1)+r_xr_y(0)+r_xr_z(0)$$
$$+r_y^2(1)+r_xr_y(0)+r_yr_z(0)$$
$$+r_z^2(1)+r_xr_z(0)+r_yr_z(0)$$

$$\left(|\vec{r}|\right)^2=r_x^2(1)+r_y^2(1)+r_z^2(1)$$

$$|\vec{r}|=\sqrt{r_x^2+r_y^2+r_z^2}$$

This concludes the proof.

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Magnetic Dipole Torque Theorem Proof for Polar Curves

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12/10/02024

Strange Cancellation Theorem

I personally do not want to call this a "theorem"; rather, I would have preferred to have called this a lemma. Nevertheless, it seems suspect. If it did not become zero, then it would not have seen suspect to my peers or myself. For those that doubt this (I made this to convince myself mainly), this might be of aid.

A crazier argument is that since the magnetic dipole theorem is true in real life, and the only way it can be in agreement with the definition for magnetic force is for those integrals, or the sum of those integrals when appearing in a sum to equal 0, then it must be. (This is a fallacious argument as we are not 100% we live in a mathematical theorem, but it is fascinating and funny.)

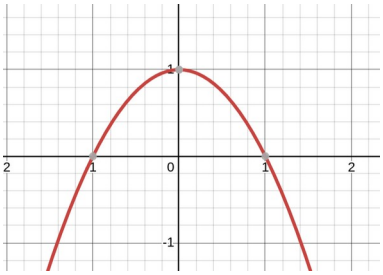
Here's a more rigorous proof:

$$Let \quad u=g(x)$$
$$\int_a^b f(g(x))g'(x)dx=\int_a^b f(u)\frac{du}{dx}dx=\int_{g(a)}^{g(b)} f(u)du$$

So, if g(a) = g(b) = G

$$Let \quad u=g(x)$$
$$\int_a^b f(u)\frac{du}{dx}dx=\int_{g(a)}^{g(b)} f(u)du=\int_G^G f(u)du=G$$ { { - 1 ^ 2 + { 1 over 2 } } 1 ^ 4 }

The following example is not a proof, but it will illustrate if things still feel "sus".



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Consider $u=1-x^2$, *we shall integrate within the interval* $1=[-1,1]$
$-2x(1-x^2)$:
According the theorem:
Let $u=g(x)=1-x^2$
Then $\frac{du}{dx}=-2x$
$g(-1)=1-(-1)^2=1-1=0$
$g(1)=1-1=0$

$$\int_{-1}^{1} -2x(1-x^2)\,dx=\int_{-1}^{1} \frac{du}{dx}(1-x^2)\,dx=\int_{0}^{0} u\,du=0$$

Let us attempt this without the theorem.

$$\int_{-1}^{1} -2x(1-x^2)\,dx=\int_{-1}^{1} -2x+2x^3\,dx=-x^2+\frac{1}{2}x^4\Big|_{-1}^{1}$$
$$=\left(-1^2+\frac{1}{2}1^4\right)-\left(-(-1)^2+\frac{1}{2}(-1)^4\right)$$
$$=\left(-1^2+\frac{1}{2}1^4\right)-\left(-1^2+\frac{1}{2}1^4\right)$$
$$=0$$

The point can become belabored with the definition of integrals or any function; however, I really don't want to do that.

Graveyard

There are mistakes but principles among them. As much as this proof can be daunting, I wanted to leave this to show that math is"non-linear" as a whole. Your journey will have ups and downs. You will make mistakes. Just find ways to check them and not only move on, but realize that it's normal no matter how weird the math gets.

Explore possibilities, accept that they can fail. Make educated guesses, do experiments to prove them.

Math is indeed the purest science ;)

The right answers are not the most important answers.

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$$\vec{r}\,'(\theta)=\left(r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)\right)\vec{i_\theta}+\left(r'(\theta)\sin(\theta)+r(\theta)\cos(\theta)\right)\vec{j_\theta}$$

Assign the functions to their corresponding component.

$$\vec{r}\,'(\theta)=r'_x(\theta)\vec{i_\theta}+'r_y(\theta)\,\vec{j_\theta}$$

Express the component vectors in bracket form.

$$\vec{r}\,'(\theta)=r'_x(\theta)\langle i_{\theta x}, i_{\theta y}\rangle+r'_y(\theta)\langle j_{\theta x}, j_{\theta y}\rangle$$

$$\vec{r}\,'(\theta)=\langle r'_x(\theta)\left(i_{\theta x}+j_{\theta x}\right), r'_y(\theta)\left(i_{\theta y}+j_{\theta y}\right)\rangle$$

Oh boy.

$$|\vec{r}\,'(\theta)|=\sqrt{r'_x(\theta)^2\left(i_{\theta x}+j_{\theta x}\right)^2+r'_y(\theta)^2\left(i_{\theta y}+j_{\theta y}\right)^2}$$

$$|\vec{r}\,'(\theta)|=\sqrt{r'_x(\theta)^2\left(i_{\theta x}^2+2\,i_{\theta x}\,j_{\theta x}+j_{\theta x}^2\right)+r'_y(\theta)^2\left(i_{\theta y}^2+2\,i_{\theta y}\,j_{\theta y}+j_{\theta y}^2\right)}$$

$$|\vec{r}\,'(\theta)|=\sqrt{r'_x(\theta)^2 i_{\theta x}^2+2\,r'_x(\theta)^2 i_{\theta x}\,j_{\theta x}+r'_x(\theta)^2 j_{\theta x}^2+r'_y(\theta)^2 i_{\theta y}^2+2\,r'_y(\theta)^2 i_{\theta y}\,j_{\theta y}+r'_y(\theta)^2 j_{\theta y}^2}$$

$$|\vec{r}\,'(\theta)|=\sqrt{\left(r'_x(\theta)^2 i_{\theta x}^2+r'_y(\theta)^2 i_{\theta y}^2\right)+\left(r'_x(\theta)^2 j_{\theta x}^2+r'_y(\theta)^2 j_{\theta y}^2\right)+\left(2\,r'_x(\theta)^2 i_{\theta x}\,j_{\theta x}+2\,r'_y(\theta)^2 i_{\theta y}\,j_{\theta y}\right)}$$

F!@#$

This can be broken down into three sections. Consider the 1st term:

$$\left(r'_x(\theta)^2 i_{\theta x}^2+r'_y(\theta)^2 i_{\theta y}^2\right)=\left(r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)\right)^2\left(i_{\theta x}^2\right)+\left(r'(\theta)\sin(\theta)+r(\theta)\cos(\theta)\right)^2\left(i_{\theta y}^2\right)$$

$$=\left(r'(\theta)^2\cos^2(\theta)-2\mathrm{r}(\theta)r'(\theta)\cos(\theta)\sin(\theta)+r(\theta)^2\sin^2(\theta)\right)\left(i_{\theta x}^2\right)$$
$$+\left(r'(\theta)^2 sin^2(\theta)+2\mathrm{r}(\theta)r'(\theta)\cos(\theta)\sin(\theta)+r(\theta)^2 cos^2(\theta)\right)\left(i_{\theta y}^2\right)$$

$$=i_{\theta x}^2 r'(\theta)^2\cos^2(\theta)-2\mathrm{i}_{\theta x}^2 r(\theta)r'(\theta)\cos(\theta)\sin(\theta)+i_{\theta x}^2 r(\theta)^2\sin^2(\theta)$$
$$+\left(i_{\theta y}^2 r'(\theta)^2\sin^2(\theta)+2\,i_{\theta y}^2 r(\theta)r'(\theta)\cos(\theta)\sin(\theta)+i_{\theta y}^2 r(\theta)^2\cos^2(\theta)\right)$$

$$=2\mathrm{i}_{\theta y}^2 r(\theta)r'(\theta)\cos(\theta)\sin(\theta)-2\mathrm{i}_{\theta x}^2 r(\theta)r'(\theta)\cos(\theta)\sin(\theta)$$
$$+i_{\theta x}^2 r'(\theta)^2\cos^2(\theta)+i_{\theta x}^2 r(\theta)^2\sin^2(\theta)$$
$$+i_{\theta y}^2 r'(\theta)^2\sin^2(\theta)+i_{\theta y}^2 r(\theta)^2\cos^2(\theta)$$

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Regroup

$$=2i_{\theta y}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)-2i_{\theta x}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)$$
$$+i_{\theta x}^2 r'(\theta)^2\cos^2(\theta)+i_{\theta y}^2 r'(\theta)^2\sin^2(\theta)$$
$$+i_{\theta x}^2 r(\theta)^2\sin^2(\theta)+i_{\theta y}^2 r(\theta)^2\cos^2(\theta)$$

$$=2i_{\theta y}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)-2i_{\theta x}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)$$
$$+i_{\theta x}^2 r'(\theta)^2\cos^2(\theta)+i_{\theta y}^2 r'(\theta)^2\sin^2(\theta)$$
$$+i_{\theta x}^2 r(\theta)^2\sin^2(\theta)+i_{\theta y}^2 r(\theta)^2\cos^2(\theta)$$

$$=i_{\theta x}^2 r'(\theta)^2\cos^2(\theta)-2i_{\theta x}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)+i_{\theta x}^2 r(\theta)^2\sin^2(\theta)$$
$$+i_{\theta y}^2 r'(\theta)^2\sin^2(\theta)+2i_{\theta y}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)+i_{\theta y}^2 r(\theta)^2\cos^2(\theta)$$

Consider the 2nd term

$$\left(r'_x(\theta)^2 j_{\theta x}^2+r'_y(\theta)^2 j_{\theta y}^2\right)=\left(r'(\theta)\cos(\theta)-r(\theta)\sin(\theta)\right)^2\left(j_{\theta x}^2\right)+\left(r'(\theta)\sin(\theta)+r(\theta)\cos(\theta)\right)^2\left(j_{\theta y}^2\right)$$

It looks like the 1st term, but instead of i, it involves j. This, indeed, shall be "left as an exercise to the reader". (Or you can read the procedure above, but pretend the i's are just really straight j's.)

Therefore, it expands to this term.

$$=2j_{\theta y}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)-2j_{\theta x}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)$$
$$j_{\theta y}^2 r'(\theta)^2\sin^2(\theta)+j_{\theta x}^2 r'(\theta)^2\cos^2(\theta)$$
$$+j_{\theta x}^2 r(\theta)^2\sin^2(\theta)+j_{\theta y}^2 r(\theta)^2\cos^2(\theta)$$

Finally, the 3rd term.

$$\left(2r'_x(\theta)^2 i_{\theta x} j_{\theta x}+2r'_y(\theta)^2 i_{\theta y} j_{\theta y}\right)=¿$$

Consider one of the inner portions:

Therefore:

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$$\left(2r'_x(\theta)^2 i_{\theta x} j_{\theta x} + 2r'_y(\theta)^2 i_{\theta y} j_{\theta y}\right) = \left(2 i_{\theta x} j_{\theta x}\left(r'(\theta)^2 + r(\theta)^2\right)\right)$$

So nothing is canceling out, yet, there is a surprise tool.

We shall establish this lemma. It means that if this complicated term results from the sum, then it will all simplify two terms, which when square rooted will imply a simpilficaiton in the main problem.

Let us sum the three terms.

$$\begin{aligned}
&j_{\theta y}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta) - j_{\theta x}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)\\
&\quad j_{\theta y}^2 r'(\theta)^2\sin^2(\theta) + j_{\theta x}^2 r'(\theta)^2\cos^2(\theta)\\
&\quad + j_{\theta x}^2 r(\theta)^2\sin^2(\theta) + j_{\theta y}^2 r(\theta)^2\cos^2(\theta)\\
&+ i_{\theta y}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta) - i_{\theta x}^2 r(\theta) r'(\theta)\cos(\theta)\sin(\theta)\\
&\quad i_{\theta y}^2 r'(\theta)^2\sin^2(\theta) + i_{\theta x}^2 r'(\theta)^2\cos^2(\theta)\\
&\quad + i_{\theta x}^2 r(\theta)^2\sin^2(\theta) + i_{\theta y}^2 r(\theta)^2\cos^2(\theta)\\
&\qquad + ¿
\end{aligned}$$

ERROR ERROR

$$\vec{r} = r_x\hat{u} + r_y\hat{v} + r_z\hat{w} = \langle r_x(u_x + v_x + w_x), r_y(u_y + v_y + w_y), r_z(u_z + v_z + w_z)\rangle$$

$$|\vec{r}| = \sqrt{r_x^2(u_x + v_x + w_x)^2 + r_y^2(u_y + v_y + w_y)^2 + r_z^2(u_z + v_z + w_z)^2}$$

$$\begin{aligned}
(u_x + v_x + w_x)^2 &= \left(u_x(u_x + v_x + w_x) + v_x(u_x + v_x + w_x) + w_x(u_x + v_x + w_x)\right)\\
&= \left(\left(u_x^2 + u_x v_x + u_x w_x\right) + \left(v_x u_x + v_x^2 + v_x w_x\right) + \left(u_x w_x + v_x w_x + w_x^2\right)\right)\\
&= \left(\left(u_x^2 + u_x v_x + u_x w_x\right) + \left(u_x v_x + v_x^2 + v_x w_x\right) + \left(u_x w_x + v_x w_x + w_x^2\right)\right)
\end{aligned}$$

$$\begin{aligned}
\hat{w} &= \langle u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x\rangle\\
&\quad = u_x^2 + u_x v_x + u_x(u_y v_z - u_z v_y)\\
&\quad + u_x v_x + v_x^2 + v_x(u_y v_z - u_z v_y)\\
&+ u_x(u_y v_z - u_z v_y) + v_x(u_y v_z - u_z v_y) + (u_y v_z - u_z v_y)^2
\end{aligned}$$

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Unused Theorems

$$\hat{v}\cdot\hat{w}=\langle v_x,v_y,v_z\rangle\cdot\langle w_x,w_y,w_z\rangle=v_xw_x+v_yw_y+v_zw_z=0$$

$$\hat{u}\cdot\hat{w}=\langle u_x,u_y,u_z\rangle\cdot\langle w_x,w_y,w_z\rangle=u_xw_x+u_yw_y+u_zw_z=0$$

$$v_x\left(u_yv_z-u_zv_y\right)+v_y\left(u_zv_x-u_xv_z\right)+v_z\left(u_xv_y-u_yv_x\right)=\left(u_yv_xv_z-u_zv_y\right)+\left(u_zv_xv_y-u_xv_yv_z\right)+\left(u_xv_yv_z-u_yv_xv_z\right)=0$$

$$u_yv_xv_z-u_zv_xv_y+u_zv_xv_y-u_xv_yv_z+u_xv_yv_z-u_yv_xv_z=0$$

$$u_x\left(u_yv_z-u_zv_y\right)+u_y\left(u_zv_x-u_xv_z\right)+u_z\left(u_xv_y-u_yv_x\right)=u_xu_yv_z-u_xu_zv_y+u_yu_zv_x-u_xu_yv_z\ \ +u_xu_zv_y-u_yu_zv_x$$

$$u_xu_yv_z-u_xu_zv_y+u_yu_zv_x-u_xu_yv_z+u_xu_zv_y-u_yu_zv_x=0$$

Combine them

$$u_yv_xv_z-u_zv_xv_y+u_zv_xv_y-u_xv_yv_z+u_xv_yv_z-u_yv_xv_z=0$$

$$u_xu_yv_z-u_xu_zv_y+u_yu_zv_x-u_xu_yv_z+u_xu_zv_y-u_yu_zv_x=0$$

Therefore

$$u_xu_yv_z-u_xu_zv_y+u_yu_zv_x-u_xu_yv_z\ \ +u_xu_zv_y-u_yu_zv_x=u_yv_xv_z-u_zv_xv_y+u_zv_xv_y-u_xv_yv_z+u_xv_yv_z-u_yv_xv_z$$

$$2u_xu_yv_z-u_xu_zv_y+u_yu_zv_x-u_xu_yv_z\ \ +u_xu_zv_y-u_yu_zv_x=u_yv_xv_z-u_zv_xv_y+u_zv_xv_y+u_xv_yv_z-u_yv_xv_z$$

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